(1) (a) A plane polarized EM wave with electric field given by $\vec{E} = \vec{e}_0 E_0 e^{i(kz-\omega t)}$ is incident normally on a semi-infinite slab of thickness d with its left surface at z=0 and dielectric constant $\varepsilon(\omega)/\varepsilon_0$ possessing finite conductivity.
With no assumptions regarding the smallness of $[\varepsilon(\omega)/\varepsilon_0 - 1]$, show that to *first order* in d, the electric field at point z downstream from the slab is given by

$$\vec{E} = \vec{e}_0 E_0 e^{i(kz-\omega t)}[1 + ik(\frac{\varepsilon(\omega)}{\varepsilon_0} - 1)\frac{d}{2}]$$

(10 points)

(b) Calculate the rate of power dissipation in the slab per unit area to lowest order in the thickness d.

(15 points)

(2) Consider circularly polarized EM waves propagating in the direction of a static magnetic field $\mathbf{B_0}$ in a medium consisting of N electrons/unit volume behaving as bound oscillators with a single oscillator resonance frequency ω_0 with oscillator strength =1 and damping constant γ.

(a) Show that the relationship between the refractive indices for waves of (+) and (-) circular polarization can be written as

$$n_+^2 - n_-^2 = \frac{Ne^2}{\varepsilon_0 m} \times$$

$$[\frac{1}{\omega_0^2 - \omega^2 - i\gamma\omega + eB_0\omega/m} - \frac{1}{\omega_0^2 - \omega^2 - i\gamma\omega - eB_0\omega/m}]$$

(12 points)

(b) Neglecting γ, from the above formula calculate the rotation in radians of the plane of polarization of a *linearly polarized* plane wave after propagating a length L in the medium when the magnetic field is present.

(Hint: Consider a linearly polarized wave as the coherent sum of two oppositely circularly polarized waves).

(13 points)

(3) The theory of magneto-optic effects shows that in a medium with magnetization density $\mathbf{M}$, the **D** and **E** fields are related via an anisotropic dielectric tensor $\varepsilon_{\alpha\beta}$

$$D_\alpha = \sum_\beta \varepsilon_{\alpha\beta} E_\beta$$

where $\varepsilon_{\alpha\beta} = \varepsilon_0 \delta_{\alpha\beta} + \delta\varepsilon_{\alpha\beta}$

and the $\delta\varepsilon_{\alpha\beta}$ are small and have the form:

$$\frac{\delta\varepsilon_{\alpha\beta}}{\varepsilon_0} = \begin{pmatrix} A & -iBM_z & iBM_y \\ iBM_z & A & -iBM_x \\ -iBM_y & iBM_x & A \end{pmatrix}$$

A and B are constants and M_x etc. are the Cartesian components of **M**.

(a) Show that Maxwell's Equations for a wave propagating in the medium with frequency ω can be written as the set of equations

$$\nabla^2 E_\alpha + k^2 E_\alpha = \nabla_\alpha(\nabla.E) - k^2 \sum_\beta \frac{\delta\varepsilon_{\alpha\beta}}{\varepsilon_0} E_\beta$$

where k = ω/c. (10 points)

(b) Using the outgoing Green's function formalism and the Born approximation (as in Secn. 10.2 in Jackson) show that the scattered electric field from a small volume of this material in the far field region can be written to first order in the $\delta\varepsilon_{\alpha\beta}$ as

$$\mathbf{E} \rightarrow \mathbf{E}^0 + \mathbf{A}_{sc}\frac{e^{ikr}}{r}$$

where $$\frac{\mathbf{A_{sc}.e_{sc}}}{E_0} = \frac{k^2}{4\pi}\int d^3x e^{i\mathbf{q.x}} \sum_{\alpha\beta} (e_{sc})_\alpha \frac{\delta\varepsilon_{\alpha\beta}}{\varepsilon_0}(e_0)_\beta$$

where $\mathbf{q} = \mathbf{k_0} - \mathbf{k_{sc}}$, (both $\mathbf{k_0}$ and $\mathbf{k_{sc}}$ have magnitude k) the spatial dependence of the incident electric field is given by $\mathbf{E}^{\mathbf{0}}(\mathbf{x}) = \mathbf{e}_{\mathbf{0}} E_0 e^{i\mathbf{k}_{\mathbf{0}}.\mathbf{x}}$

and $\mathbf{e}_{\mathbf{sc}}$ is one of the possible polarization vectors of the scattered beam. (10 points)

(c) Assume the incident wave is propagating along the z-direction with $\mathbf{e}_{\mathbf{0}}$ along the x-direction and **M** is parallel to the z-direction and the volume is a sphere of radius a. For scattering through an angle θ in the y-z plane, calculate $\frac{d\sigma}{d\Omega}$ explicitly for the outgoing electric field polarized in the y-z plane; and in the x-direction (i.e. perpendicular to the y-z plane). (15 points)

NOTE 1: Do NOT consider any contributions from δμ-type terms!
NOTE 2: The cross-sections cannot involve E_0

(4) Use the Kramers-Kronig relation to calculate the real part of ε(ω), given the imaginary part of ε(ω) for positive ω as

$$\mathrm{Im}(\varepsilon / \varepsilon_0) = \lambda[\theta(\omega - \omega_1) - \theta(\omega - \omega_2)]$$

$$\omega_2 > \omega_1 > 0$$

$$\theta(x) = 0, x < 0; \theta(x) = 1, x > 0$$

Sketch the behavior of Re ε(ω) and Im ε(ω) as functions of ω.

(15 points)